Signed Integer Representation Using

Two's Complement!

Intuition

The Intuition of Two's Complement Representation of Signed Integers

- Big Idea:

Negate the value of the *highest order bit*
and *treat every other bit the same*.

- Example:
  - 0b1000 to an unsigned decimal is: $\mathbf{1} \times \mathbf{2^3} \rightarrow \mathbf{8}$
  - 0b1000 to a two's complement signed decimal is: $\mathbf{-1} \times \mathbf{2^3} \rightarrow \mathbf{-8}$

Two's Complement Signed Binary Values

Suppose we define a Base-2 number b, with w place values, (*w stands for width*) as a vector of **binary digits**:

$$\vec{b} = [b_{w-1}, b_{w-2}, \dots, b_0]$$

We can determine the value of $\vec{d}$ with the following equation:

$$Binary2TwosComplement_w\left(\vec{b}\right) = -b_{w-1}2^{w-1} + \sum_{i=0}^{w-2} b_i 2^i$$

A concrete example:

$$w = 4$$

$$\vec{b} = [1, 1, 1, 0]$$

thus

$$b_3 = 1$$
$$b_2 = 1$$
$$b_1 = 1$$
$$b_0 = 0$$

$$Binary2TwosComplement_4([1, 1, 1, 0])$$

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$$Binary2TwosComplement_4([1, 1, 1, 0])$$

$$= -b_{w-1}2^3 + \sum_{i=0}^{2} b_i 2^i$$

$$= -b_3 \times 2^3 + b_0 \times 2^0 + b_1 \times 2^1 + b_2 \times 2^2$$

$$= -1 \times 2^3 + 0 \times 2^0 + 1 \times 2^1 + 1 \times 2^2$$

$$= -8 + 0 + 2 + 4$$

$$= -2$$

Negation of a Two's Complement Value

- To negate any signed-value with Two's Complement representation:
  1. Take the complement (~) of all bits (0s -> 1s, 1s -> 0s)
  2. Add 1

$$\vec{b} = [0, 0, 0, 1] = 1$$

$$\sim\vec{b} = [1, 1, 1, 0] = -2$$

$$\sim\vec{b} + [0,0,0,1] = [1, 1, 1, 1] = -1$$

$$\vec{b} = [1, 1, 1, 1] = -1$$

$$\sim\vec{b} = [0, 0, 0, 0] = 0$$

$$\sim\vec{b} + [0,0,0,1] = [0, 0, 0, 1] = 1$$

https://www.cs.cornell.edu/~tomf/notes/cps104/twoscomp.html

Interesting Properties of Two's Complement

- Place values $\boldsymbol{w-2 \ldots 0}$ are ***the same*** regardless of signed or unsigned
  - *No steps needed to convert between the two!*
  - *The only difference is how you interpret the high-order bit.*

- Moving a signed number to a larger memory width only requires extending the left most bit to fill in the gap
  - Negative numbers fill left place values with 1s, positive with 0s.
  - This is called *sign extension.*

- Subtraction works the same for both signed and unsigned values: negate the subtracted value by taking the two's complement and then just use addition!

- Trade-off: Off-by-one asymmetry in negative versus positive range
  - $-2^{w-1}\ \ to\ 2^{w-1}-1$ - one extra negative number compared with positive